C3.Q203.CH16A.LESSON1

LINE INTEGRALS

Consider a smooth curve C:

$\int_C ds$ = length of curve (or wire) C

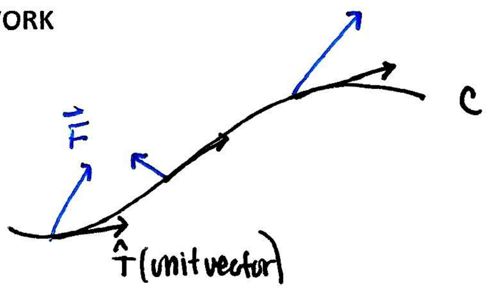
$\int_C f(x(t), y(t), z(t))\,ds$: mass of curve C where $f(t)$ is density function

$\int_C \vec{F}\cdot d\vec{r}$ = work done by force F or vector field $\vec{F}$ as it moves a particle along curve C (in positive orientation)

VECTOR FIELD

$$\vec{F} = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$$
$$= P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$$
$$= \langle P, Q, R \rangle$$

WORK



Increments of work $= \vec{F}\cdot\hat{T}$

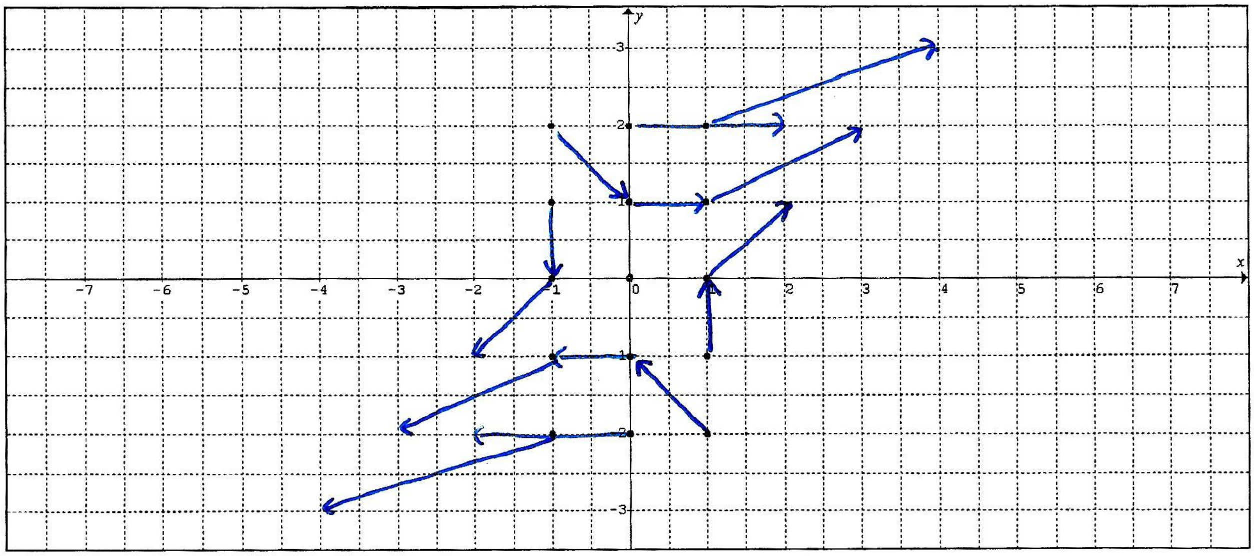
$$\text{Total work} = \overset{\text{[CONCEPT]}}{\underbrace{\int_C \vec{F}\cdot\hat{T}\,ds}_{\text{summing increments along path}}} = \overset{\text{[DERIVATION]}}{\int_C \vec{F}\cdot \underbrace{\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}_{\hat{T}} \underbrace{|r'(t)|\,dt}_{ds}} = \overset{\text{[COMPUTATION]}}{\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt} = \overset{\text{[NOTATION]}}{\int_C \vec{F}\cdot d\vec{r}}$$

$$= \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle$$

[NON-VECTOR]

$$= \int_C (P\,dx + Q\,dy + R\,dz)$$

GRAPH THE VECTOR FIELD $\mathbf{F}(x,y) = \langle x+y,\, x \rangle$ for the given 15 points shown below:

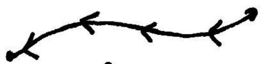


Orientation of a Non-Closed Line Integral

$C: a \le t \le b \qquad a < b \qquad C(a) \ne C(b)$

Positive orientation: $\displaystyle\int_C f\,ds = \int_a^b f\,ds$

Negative orientation: $\displaystyle\int_{-C} f\,ds = \int_b^a f\,ds$

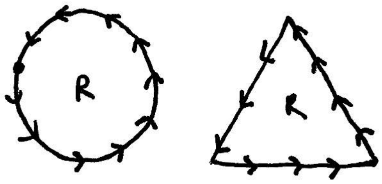


facing in opposite direction

Orientation of a Closed Line Integral

$C: D: a \le t \le b \qquad a < b \qquad c(a) = c(b)$

$t = a = b$



Positive orientation: CCW

area enclosed is to your left if walking along C with head in $+z$

Mass: $\displaystyle\oint_C f\,ds$

As always: $\displaystyle\oint_C = -\oint_{-C}$

Work: $\displaystyle\oint_C P\,dx + Q\,dy + R\,dz = \oint_C \vec{F} \cdot d\vec{r}$

C3.Q203.CH16A.LESSON2

CONSERVATIVE VECTOR FIELD

DEFINITION: Vector field $\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some scalar function f

f is the potential function for $\vec{F}$

Is **F** conservative? Methods to Determine:

1. Find f such that $\nabla f = \mathbf{F}$.

   A. "Magically" Pick f.

   B. Use "JAY'S METHOD" to find f.

2. Use a Theorem. **F** is conservative if ...

   A. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in $\Re^2$

   B. $curl\mathbf{F} = \vec{\mathbf{0}}$ in $\Re^2$ and in $\Re^3$

EXAMPLES: Determine if **F** is conservative.

① $\vec{F} = \langle \underbrace{4xyz + 6}_{P},\ \underbrace{2x^2z - 5}_{Q},\ \underbrace{2x^2y + 4}_{R} \rangle$

A. magically: $f(x,y,z) = 2x^2yz + 6x - 5y + 4z$

verify: $\nabla f(x,y,z) = \langle 4xyz+6,\ 2x^2z-5,\ 2x^2y+4 \rangle = \vec{F}$ ✓

$\therefore$ $\vec{F}$ is conservative

B. Jay's method:

$$\int P\,dx = \int (4xyz + 6)\,dx = 2x^2yz + 6x + k(y,z)$$

$$\int Q\,dy = \int (2x^2z - 5)\,dy = 2x^2yz - 5y + k(x,z)$$

$$\int R\,dz = \int (2x^2y + 4)\,dz = 2x^2yz + 4z + K(x,y)$$

Factors into constant

$\int P\,dx$ cannot have terms that do not have x

etc.

$$f(x,y,z) = \underbrace{2x^2yz}_{\text{write if common between}} + \underbrace{6x - 5z + 4z}_{\text{write if they're singletons}} + c$$

$\therefore$ f is conservative

C. $\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = i(2x^2 - 2x^2) + j(4xy - 4xy) + k(4xz - 4xz)$

$\boxed{\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle}$

$= \langle 0,0,0 \rangle = 0$

$\therefore$ f is conservative

② $\vec{F} = \langle 3x^2y - xy + 2 \, , \, x^3 + xy \rangle$

A. <u>magically</u> : none possible ??!#@3*!

B. <u>mathically</u> : $\int P\,dx = \int (3x^2y - xy + 2)\,dx = x^3y - \frac{1}{2}x^2y + 2x + k(y)$

good   different

$\int Q\,dx = \int (x^3 + xy)\,dy = x^3y + \frac{1}{2}xy^2 + k(x)$

<u>If</u> : $f(x,y) = x^3y + \frac{1}{2}xy^2 - \frac{1}{2}x^2y + 2x + C$

$\nabla f(x,y) \neq \vec{F}$ $\quad \therefore$ F is not conservative

C. $\frac{\partial P}{\partial y} \stackrel{?}{=} \frac{\partial Q}{\partial x} \rightarrow \frac{\partial}{\partial y}(3x^2y - xy + 2) = 3x^2 - x$

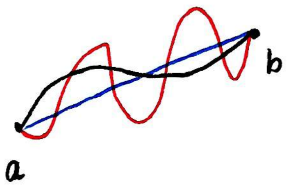
$\rightarrow \frac{\partial}{\partial x}(x^3 + xy) = 3x^2 + y$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ $\quad \therefore$ F is not conservative

FOUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= \underbrace{f(x_2, y_2, z_2)}_{\text{potential at b}} - \underbrace{f(x_1, y_1, z_1)}_{\text{potential at a}}$$



work done is same along all paths.

EXAMPLE:

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2xz + y^2, 2xy, x^2 + 3z^2 \rangle$ and $C: x = t^2,\ y = t + 1,\ z = 2t - 1,\ 0 \le t \le 1$

<u>Step 1</u>: determine if f is conservative by <u>Jay's method</u>

$$\int P\,dx = \int (2xz + y^2)\,dx = x^2 z + xy^2 + k(y,z)$$

$$\int Q\,dy = \int (2xy)\,dy = xy^2 + k(x,z)$$

$$\int R\,dz = \int (x^2 + 3z^2)\,dz = x^2 z + z^3 + k(x,y)$$

Potential $= f = xy^2 + xz^2 + z^3 + C \quad \therefore$ F is conservative

<u>Step 2</u>: Find a and b:

$t = 0 \rightarrow x = 0^2 = 0$
$\quad y = 0 + 1 = 1$
$\quad z = 2 \cdot 0 - 1 = -1$

$t = 1 \rightarrow x = 1$
$\quad y = 2$
$\quad z = 1$

<u>Step 3</u>: use FTC for line integrals

$$f(1,2,1) - f(0,1,-1) = (1 \cdot 2^2 + 1 \cdot 1^2 + 1^3) - (0 + 0 + (-1)^3)$$

$$= 6 - (-1) = \boxed{7}$$

IF AND ONLY IF … THE 99.9%

Takeaway:

$$\left.\begin{array}{l} \vec{F} = \nabla \vec{f} \\ \text{curl } \vec{F} = 0 \\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \end{array}\right\} \rightarrow \left\{\begin{array}{l} \rightarrow \vec{F} \text{ is conservative} \\ \rightarrow \int \vec{F} \cdot d\vec{r} \text{ independent of path} \\ \rightarrow \text{use FTC} \end{array}\right.$$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

closed line integral
start and end at same point

$f(\text{same}) - f(\text{same}) = 0$

IN CLASS PRACTICE: 16.3 # 1, 22, 12, 8, 20

C3.Q203.CH16A.LESSON3

ALL ABOUT DEL

$\vec{F} = \langle P, Q, R \rangle$

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f(x,y,z) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\nabla \times \vec{F} = \text{curl } \vec{F}$ → provides indication about rotational aspects of motion

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

In R^2: curl $\vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$ → set $= 0$ to test for conservativity

$\nabla \cdot \vec{F} = \text{div } \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \langle \frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z} \rangle$

→ Flow of mass (divergence of mass)

→ Is more mass flowing towards or away?

Prove: If **F** is conservative, then *curl* $\mathbf{F} = \vec{\mathbf{0}}$

$\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some scalar function f

$$\vec{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\text{curl } \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left\langle \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)}_{0}, \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)}_{0}, \underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)}_{0} \right\rangle$$

By CLAIRAUT'S THEOREM

$= \langle 0, 0, 0 \rangle \quad \therefore \text{ curl } \vec{F} = \vec{0}$ QED

Prove: *div curl* $\mathbf{F} = 0$

$$\text{div curl } \vec{F} = \nabla \cdot (\nabla \times \vec{F})$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

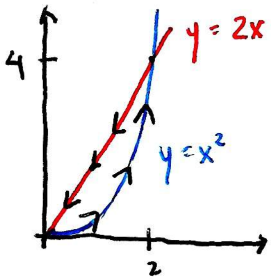
GREEN'S THEOREM (R^2)

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

for region R enclosed by positively oriented C

EXAMPLES:

1. Evaluate $\oint_C \overbrace{\underbrace{5xy}_{P}\,dx + \underbrace{x^3}_{Q}\,dy}^{\text{work}}$ where C is the positively oriented closed curve consisting of $y = x^2$ and $y = 2x$ between points (0, 0) and (2, 4).



$P = 5xy \qquad Q = x^3$

$\frac{\partial P}{\partial y} = 5x \qquad \frac{\partial Q}{\partial x} = 3x^2$

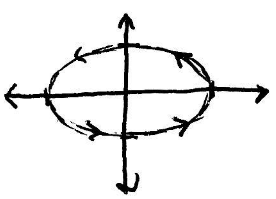
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - 5x$

$$\oint_C (5xy\,dx + x^3 dy) = \int_0^2 \int_{x^2}^{2x} (3x^2 - 5x)\,dy\,dx$$

$$= \int_0^2 \left[3x^2y - 5xy \right]_{x^2}^{2x} dx = \int_0^2 (11x^3 - 10x^2 - 3x^4)\,dx$$

$$= \left[\frac{11}{4}x^4 - \frac{10}{3}x^3 - \frac{3}{5}x^5 \right]_0^2 = -\frac{28}{15}$$

2. Evaluate $\oint_C 2xy\,dx + (x^2 + y^2)\,dy$ where C is the positively oriented ellipse $4x^2 + 9y^2 = 36$



$P = 2xy \qquad Q = x^2 + y^2$

$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

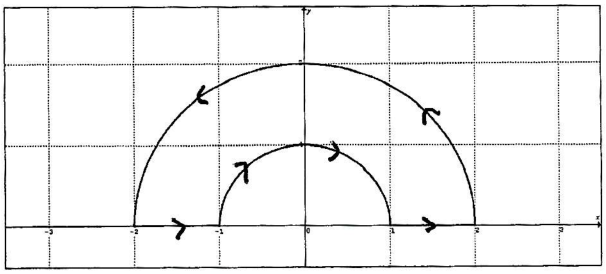
* NOTE that

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \rightarrow F$ is conservative

$\oint_C (\text{stuff}) = 0$

$$\oint_C (2xy\,dx + (x^2 + y^2)\,dy) = \iint_R 0\,dA$$

3. Evaluate $\oint_C y^2dx + 3xydy$ where C is outline of the the half circular washer with the positive orientation as shown in the graph below:



$P = y^2 \qquad Q = 3xy$

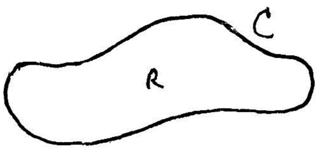
$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 3y \qquad T: \; x = r\cos\theta, \; y = r\sin\theta$

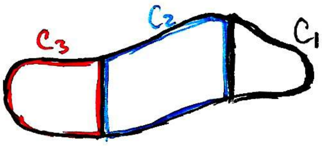
$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2y = y = r\sin\theta$

$= \int_0^{\pi}\int_1^{2} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{\pi}\sin\theta\,d\theta \int_1^2 r^2\,dr$

$= \left[-\cos\theta\right]_0^{\pi}\left[\frac{1}{3}r^3\right]_1^2 = 2\cdot\frac{7}{3} = \frac{14}{3}$

EXPLAIN THIS ...





$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1}\vec{F}\cdot d\vec{r} + \oint_{C_2}\vec{F}\cdot d\vec{r} + \oint_{C_3}\vec{F}\cdot d\vec{r}$$

VECTOR FORM OF GREEN'S THEOREM

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \text{curl}\,\vec{F}\cdot\vec{k}\,dA$$

Note that $\text{curl}\,\vec{F}\cdot\vec{k} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle \cdot \langle 0,0,1\rangle = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

C3.Q203.CH16A.LESSON4

$\boxed{2}$ **THEOREM** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$\boxed{3}$ **THEOREM** $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

$\boxed{4}$ **THEOREM** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

$\boxed{5}$ **THEOREM** If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

reverse

$\boxed{6}$ **THEOREM** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

$\boxed{4}$ **THEOREM** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

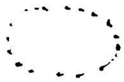
GREEN'S THEOREM Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Curve C is simple: does not intersect itself at endpoints

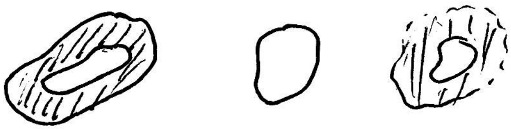
Curve C is closed: endpoints are equal

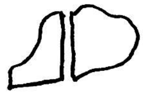
think closed interval

Region R is open:  doesn't include boundary

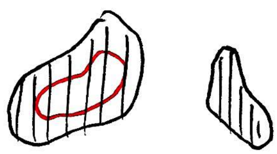
Region R is closed:  includes boundary

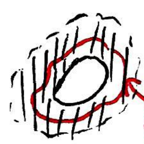
Region R is connected: any two points in R can be connected with a path within D



NOT connected: 

Region R is simply-connected: every simple closed curve in D encloses only points in D



NOT simply-connected: 

encloses points not in D

SECTION 16.3 #29 - 30

The Special Problem (The 0.01%)

Consider the vector field $\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{+x}{x^2+y^2}\mathbf{j}$ **(Special Problem)**

A. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is any curve that does not enclose the origin. (because not in domain)

What method did you use and why were you able to use it?

B. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is a circle of radius "a" that does enclose the origin.

What method did you use and why were you able to use it?

C. Prove that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any curve C that encloses the origin.

A. $P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{-x}{x^2+y^2}$

We don't know if it's conservative, but we can still use Green's Theorem!

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(+1) - (+x)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0 \, dA = 0$$

B. $\oint_C \vec{F}\cdot d\vec{r}$

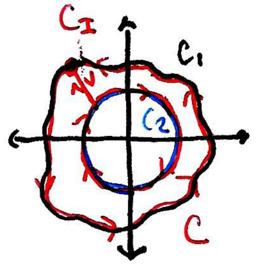
$\vec{r}(t) = \langle a\cos t, a\sin t\rangle \quad 0 \le t \le 2\pi \rightarrow d\vec{r} = \langle -a\sin t, a\cos t\rangle dt$

$\vec{F}(t) = \langle \frac{-1}{a}\sin t, \frac{1}{a}\cos t\rangle$

$\vec{F}(t)\cdot d\vec{r} = -\frac{1}{a}\sin t \cdot -a\sin t + \frac{1}{a}\cos t \cdot a\cos t = 1$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 1\, dt = [t]_0^{2\pi} = 2\pi$

C.



$\oint_C \vec{F}\cdot d\vec{r} = 0$ where C does not enclose the origin

$$= \underbrace{\oint_{C_1} \vec{F}\cdot d\vec{r}}_{2\pi} + \cancel{\oint_{C_I} \vec{F}\cdot d\vec{r}}^{\,0} + \underbrace{-\oint_{-C_2} \vec{F}\cdot d\vec{r}}_{0} + \cancel{\oint_{C_I} \vec{F}\cdot d\vec{r}}^{\,0}$$

$\therefore \oint_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$ where C_1 is any closed curve that encloses the origin

Can we use Green's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}$ defined in section 16.3 #9 and 21?

PLEASE NOW WATCH AND ENJOY THE CH16A ONLINE SPRINKLES: VIDEO LINK WILL BE PROVIDED

- ☐ What is an inverse square field?
- ☐ Why is an inverse square field called as such?
- ☐ Show that an inverse square field is conservative.
- ☐ Prove $\oint_C xdy - ydx$ gives the area of region R enclosed by its boundary C.
- ☐ Prove that the area of a circle with radius "a" is πa^2

HW:

WATCH ONLINE VIDEO SPRINKLES

SECTION 16.5: #9 – 11, 12

SECTION 16.3: #23, 24